APPLICATIONS OF H – FUNCTION IN STATISTICAL DISTRIBUTION THEORY

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***Abstract*—The H- function is a special function which has wide applications in the field of statistical distribution . By using the definitions and its properties one can easily express most of the statistical distributions in terms of H- function particularly for a continuous random variable. In this paper we attempt to prove the same results of mean , variance and moment generating functions of Gamma distributions by using Mellin- BARNES Integral , properties of Gamma function and H- function**

Keywords— H- function , Gamma function , MELLIN-BARNES Integral , Gamma Distribution , Mean , Variance

Introduction

H function is denoted by $H(\boldsymbol{x}) = H_{p,\ q}^{m\ ,n}\ (z)$ [1] , [2] and which can be defines in two different forms

$$H(z) = H_{p,\ q}^{m\ ,n}\left[z\,/\ \left(a_p\,,A_p\right)\ :\ \left(b_q\,,B_q\right)\ \right]$$

$$= H_{p,\ q}^{m\ ,n}\begin{bmatrix} z\ \ /\left(a_p\,,A_p\right) \\ \left(b_q\,,B_q\right)\end{bmatrix}$$

$$H_{p,\ q}^{m\ ,n}\begin{bmatrix} z & /\,(a_1\,,A_1)\,,---(a_n\,,A_n),\ (a_{n+1}\,,A_{n+1})---\ (a_p\,,A_p) \\ & /\ \ (b_1\,,B_1)\,,---(b_m\,,B_m),\ (b_{m+1}\,,B_{m+1})---\ (b_q\,,B_q)\end{bmatrix}$$

$$= \frac{1}{2\pi\, i}\int_{L_1} \theta(s)\ z^{-s}\ ds$$

$$\text{Where}\quad \theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\ \prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\ \prod_{j=n+1}^{p}\Gamma(a_j + A_j s)}.$$

here A_j and B_j are positive real numbers and z and all a_j and b_j may be real or complex numbers . Also the parameters m, n, p, q $\in$ Z such that $0 \le m \le q$ and $0 \le n \le p$. Note that empty product forms unity . L_1 is a contour in the complex plane which runs from $c - i\infty$ to $c + i\infty$ such that all the poles $\Gamma\left(b_j + B_j s\right)$, j = 1 , 2 - - - m lies to the left of L_1 and the poles $\Gamma\left(1 - a_j - A_j s\right)$, j = 1,2 - - - n lies to the right of L_1

Alternatively H function can be defined as follows

$$H(z) = H_{p,\ q}^{m\ ,n}\left[z\,/\ \left(a_p\,,A_p\right)\ :\ \left(b_q\,,B_q\right)\ \right]$$

$$= H_{p,\ q}^{m\ ,n}\begin{bmatrix} z\ /\ \left(a_p\,,A_p\right) \\ \left(b_q\,,B_q\right)\end{bmatrix}$$

$$= H_{p,\ q}^{m\ ,n}\begin{bmatrix} z & /\ \ (a_1\,,A_1)\,,---(a_n\,,A_n),\ (a_{n+1}\,,A_{n+1})---\ (a_p\,,A_p) \\ & /\ \ (b_1\,,B_1)\,,---(b_m\,,B_m),\ (b_{m+1}\,,B_{m+1})---\ (b_q\,,B_q)\end{bmatrix}$$

$$= \frac{1}{2\pi\, i}\int_{L_2} \theta^*(s)\ z^{+s}\ ds$$

$$\text{Where}\quad \theta^*(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\ \prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + B_j s)\ \prod_{j=n+1}^{p}\Gamma(a_j - A_j s)}$$

Here also L_2 is a contour in the complex plane which runs from $c - i\infty$ to $c + i\infty$ such that all the poles $\Gamma\left(b_j - B_j s\right)$, j = 1 , 2 - - - m lies to the right of L_2 and the poles $\Gamma\left(1 - a_j + A_j s\right)$, j = 1,2 - - - n lies to the left of L_2 .

A . Preliminary Results in the Literature

In this section some important results were presented in the literature given without proof

1. **Pochchammer Symbol** :- It is denoted by $(\alpha)_r$

$$\therefore\ (\alpha)_r = \alpha\,(\alpha+1)(\alpha+2)\ .\ .\ .(\alpha+r-1)\ ,$$

$(\alpha)_0 = 1 \quad , \quad \alpha \neq 0$

2 .Gamma Function :- It is denoted by $\Gamma(\alpha)$

(a) $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

(b) $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$

(c) $\int_0^\infty x^{\alpha-1} e^{-ax}\, dx = \frac{\Gamma(\alpha)}{a^\alpha}$

3. Mellin - Barnes Integral for an Exponential function [5]

(i) $f(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\, z^{-s}\, ds = e^{-z}$

$\Rightarrow e^{-z} = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} z^v = H_{1,\ 1}^{1,\ 0}\left[z \ / \ \overline{\quad} \ (0,1) \right]$

(ii) $H_{1,\ 1}^{1,\ 0}\left[z \ / \ \overline{\quad} \ (b,1) \right] = z^b e^{-z}$

(iii) $f(z) = \frac{1}{2\pi i} \frac{1}{\Gamma(a)} \int_{c-i\infty}^{c+i\infty} \Gamma(-s)\, \Gamma(s+a)\, (-z)^s\, ds = (1-z)^{-a}$

$\Rightarrow (1-z)^{-a} = \sum_{v=0}^{\infty} \frac{(a)_v}{v!} z^v = H_{1,\ 1}^{1,\ 1}\left[-z \ / \ \begin{matrix} (1-a,1) \\ (0,1) \end{matrix} \right]$

4 .Properties of H function :-

(a) Reciprocal argument : -

$$H_{p,\ q}^{m,n}\left[z \ / \ \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] = H_{p,\ q}^{m,n}\left[\frac{1}{z} \ / \ \begin{matrix} (1-b_q, B_q) \\ (1-a_p, A_p) \end{matrix} \right]$$

(b) Argument to a real power :-

$$H_{p,\ q}^{m,n}\left[z^k \ / \ \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] =$$

$$(-k)^{-1} \ H_{q,\ p}^{n,m}\left[z \ / \ \begin{matrix} \left(1-b_q, \frac{-B_q}{k}\right) \\ \left(1-a_p, \frac{-A_p}{k}\right) \end{matrix} \right] \text{ when } k > 0$$

$$H_{p,\ q}^{m,n}\left[z \ / \ \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] =$$

$$-k \ H_{q,\ p}^{n,m}\left[z^k \ / \ \begin{matrix} (1-b_q, -kB_q) \\ (1-a_p, -kA_p) \end{matrix} \right] \text{ when } k < 0$$

(c) Multiplication by the Argument to a power k :-

$$z^k H_{p,\ q}^{m,n}\left[z^k \ / \ \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] =$$

$$H_{p,\ q}^{m,n}\left[z^k \ / \ \begin{matrix} (a_p + kA_p, A_p) \\ (b_q + kB_q, B_q) \end{matrix} \right]$$

3. Derivatives of H function :-

The r^{th} derivative of a H function can be defined using SKIBINSKI's formula [5]

$$z^r \frac{d^r}{dx^r} H_{p,\ q}^{m,n}\left[z^k \ \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] =$$

$$H_{p+1,\ q+1}^{m,\ n+1}\left[z^k \ \begin{matrix} (0,k), (a_p, A_p) \\ (b_q, B_q)\, (r,k) \end{matrix} \right]$$

B. Main Results :-

In this section we are expressing Gamma function and its distribution in terms of H functions and H distributions hence we find the Mean , Variance, and Moment generating function using definition of H - function and hence arrive the same result as given in the literature .

C. Gamma Distribution :-

A Continuous Random Variable X is called Gamma distribution if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\Gamma(m)} a^m x^{m-1} e^{-ax}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

The above function can also be expressed in the form of H function as follows

$$f(x) = \begin{cases} \frac{a}{\Gamma(m)} H_{0,\ 1}^{1,0}\left[\begin{matrix} ax \ / & \overline{\quad} \\ & (m-1,1) \end{matrix} \right], & x \geq 0 \\ 0, & otherwise \end{cases}$$

By using gamma function and definition of H function one can derive the above result

Now consider $f(x) = \frac{a}{\Gamma(m)} H_{0,\ 1}^{1,0}\left[ax \Big/ \begin{matrix} \overline{\quad} \\ (m-1,1) \end{matrix} \right]$

$$= \frac{a}{\Gamma(m)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(m-1+1.s)\ (ax)^{-s}\, ds$$

$$= \frac{a}{\Gamma(m)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s+m-1)\ (ax)^{-(s+m-1-m+1)}\, ds$$

$$= \frac{a}{\Gamma(m)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s+m-1)\, (ax)^{-(s+m-1)}\, (ax)^{m-1}\, ds$$

$$= \frac{a}{\Gamma(m)} \frac{1}{2\pi i} (a)^{m-1} (x)^{m-1} \int_{c-i\infty}^{c+i\infty} \Gamma(s+m-1)\, (ax)^{-(s+m-1)}\, ds$$

$= \frac{a^m}{\Gamma(m)} (x)^{m-1} e^{-ax}$ which completes the proof

D. Mean and variance of Gamma distribution in terms H function :-

H function is a generalized form of any special functions which can be reduced in to any continuous distribution .

In this section we deal with mean and variance of gamma distribution using the definition of H function as special function In the statistical literature the pdf of gamma function is

$f(x) = \frac{1}{\Gamma(m)}\ a^m\ x^{m-1}\ e^{-ax}$ then its mean =

E(X) = $\int_0^{\infty}\ x\, f(x)\, dx\ =\ \frac{m}{a}$ and

Variance = V(X) = E(X^2) $-\ \ [E(X)]^2\ = \frac{m}{a^2}$ [4]

The gamma function can be expressed in the form of H – function

$$f(x) = \frac{a}{\Gamma(m)}\ H_{0,\ 1}^{1\ ,0}\left[\ \begin{matrix} ax\ / & - \\ & (m-1\ ,1) \end{matrix}\ \right] =$$

$$H(x) = \frac{a}{\Gamma(m)}\ (ax)^{m-1}\ \ e^{-ax}$$

Mean = E(X) = $\int_0^{\infty}\ x\, f(x)\, dx$

$= \int_0^{\infty}\ x\ \ H(x)\, dx$ integrating by parts)

$= \ \left\{H(x)\ \frac{x^2}{2}\ -\ \ \int_0^{\infty}\ \frac{x^2}{2}\ \ H'(x)\, dx\right\}$

$= \left\{0\ -\ \int_0^{\infty}\ \frac{x^2}{2}\ \ \frac{d}{dx}\left(\frac{a}{\Gamma(m)}\ \ (ax)^{m-1}\ \ e^{-ax}\right)\ dx\right\}$

$= \ \frac{a}{\Gamma(m)}\ .\ \frac{-1}{2}.\ (a)^{m-1}\ \ \{\ \ \int_0^{\infty}\ x^2\ \ [x^{m-1}\ \ \ e^{-ax}\ .-a\ \ +\ e^{-ax}\ (m-1)x^{m-2}\ ]\ dx\ \}$

$$= -\frac{a^m}{\Gamma(m)}\left(\frac{1}{2}\right)\left\{\ \ \int_0^{\infty}\begin{matrix} -a\ x^{(m+2)-1}\ \ e^{-ax}\ dx + \\ (m-1)\int_0^{\infty} x^{(m+1)-1}\ \ \ e^{-ax}\ dx \end{matrix}\right\}$$

$$= \ -\frac{a^m}{\Gamma(m)}\left(\frac{1}{2}\right)\left\{\ -a\ \frac{\Gamma(m+2)}{a^{m+2}}\ \ +\ (m-1)\frac{\Gamma(m+1)}{a^{m+1}}\right\}$$

$$= \ -\frac{a^m}{\Gamma(m)}\left(\frac{1}{2}\right)\left\{\ -\ \frac{\Gamma(m+2)\ +\ (m-1)\Gamma(m+1)}{a^{m+1}}\ \ \right\}$$

$$= \ -\frac{1}{\Gamma(m)}\left(\frac{1}{2}\right)\left\{\ -\ \frac{(m+1)(m)\Gamma(m)\ +\ (m-1)m\Gamma(m)}{a}\ \ \right\}$$

$$= \ -\frac{1}{\Gamma(m)}\left(\frac{1}{2}\right)\ (m)\Gamma(m)\ \left\{\frac{-m-1+m-1}{a}\ \ \right\}$$

$$= \ \frac{m}{a}$$

By similar calculations we can obtain the following result

$$E\ (X^2)\ = \int_0^{\infty}\ \ x^2\, f(x)\, dx\ \ \ \int_0^{\infty}\ \ x^2\, H(x)\, dx$$

$$= \ \ -\frac{a^m}{\Gamma(m)}\left(\frac{1}{3}\right)\left\{\ -a\ \frac{\Gamma(m+3)}{a^{m+3}}\ \ +\ (m-1)\frac{\Gamma(m+2)}{a^{m+2}}\right\}$$

$$= -\frac{a^m}{\Gamma(m)}\left(\frac{1}{3}\right)\left\{\ -\ \frac{(m+2)(m+1)(m)\Gamma(m)\ +\ (m-1)(m+1)m\Gamma(m)}{a^{m+2}}\ \ \right\}$$

$$= \frac{m(m+1)}{a^2}$$

Hence variance = $\frac{m}{a^2}$

7. Moment Generating function (MGF) of Gamma function in the form of H function :-

MGF about the origin is $M_X(e^{tx})\ = E(e^{tx})$

$$= \left(1 - \frac{t}{a}\right)^{-m}$$

Now $\ f(z) = \frac{1}{2\pi\, i}\ \frac{1}{\Gamma(a)}\ \int_{c-i\infty}^{c+i\infty}\ \ \Gamma(-s)\ \ \Gamma(s+a)\ (-z)^s$ ds

$$=\ (1-z\ )^{-a}\ \ \ =\ \sum_{v=0}^{\infty} \frac{(a)_v}{v!}\ z^v$$

$$= \ \frac{1}{\Gamma(a)}\ H_{1,\ \ 1}^{1,\ \ 1}\left[\ \begin{matrix} -z & / & (1-a\ ,1\ ) \\ & & (0\ ,1) \end{matrix}\ \right]$$

Consider $\ \frac{1}{\Gamma(a)}\ \frac{1}{2\pi\, i}\ \ \int_{c-i\infty}^{c+i\infty}\ \ \theta(s)\ \ (-z)^{-s}$ ds

$$= \frac{1}{\Gamma(a)}\ \frac{1}{2\pi\, i}\int_{c-i\infty}^{c+i\infty}\ \Gamma(0+s)\Gamma(1-(1-m)-s)\ (-z)^{-s}\ \ ds$$

$$=\ \ \frac{1}{\Gamma(a)}\ \ \frac{1}{2\pi\, i}\ \ \ \int_{c-i\infty}^{c+i\infty}\ \ \Gamma(s)\Gamma(-s+m)\ \ \ (-z)^{-s}\ \ ds$$

By putting $-\ s\ =$ s $\ \ z\ =\ \frac{t}{a}$ one can reduces in to

$$f(z) = \frac{1}{2\pi\, i}\ \frac{1}{\Gamma(a)}\ \int_{c-i\infty}^{c+i\infty}\ \ \Gamma(-s)\ \ \Gamma(s+a)\ \left(-\frac{t}{a}\right)^s\ \ ds\ \ =$$

$$= \left(1 - \frac{t}{a}\right)^{-a}$$

$$=\ \sum_{v=0}^{\infty} \frac{(m)_v}{v!}\ \left(\frac{t}{a}\right)^v$$

E. Moment about the origin

In this section , Using the above relation one can find moment about the origin , by putting v = 1 , 2 , . ..

When v = 1 , we get $\frac{(m)_1}{1!}\ \left(\frac{t}{a}\right)^1$

$$= \frac{m}{1!} . \frac{t}{a}$$

Then $\mu_1{}'$ = co efficient of $\ \frac{t}{1!}\ =\ \frac{m}{a}$ = mean

When v = 2 , we get $\frac{(m)_2}{2!}\ \left(\frac{t}{a}\right)^2$

$$= \frac{m(m+1)}{2!} . \left(\frac{t}{a}\right)^2$$

Then $\mu_2{}'$ = co efficient of $\ \frac{t^2}{2!}\ =\ \frac{\Gamma(m+2)}{a^2\ \Gamma(m)}$

$\mu_3{}'$ = co efficient of $\ \frac{t^3}{3!}\ =\ \frac{\Gamma(m+3)}{a^3\ \Gamma(m)}$

$\mu_4{}'$ = co efficient of $\ \frac{t^4}{4!}\ =\ \frac{\Gamma(m+4)}{a^4\ \Gamma(m)}$

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F. Conclusion :-

Similar idea and methodology can be applied in Beta distributions of type – I and type- II , exponential and uniform distribution also.

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